Photon-added coherent states as nonlinear coherent states

S. Sivakumar

Laboratory and Measurements Section

307, General Services Building

Indira Gandhi Centre for Atomic Research

Kalpakkam, INDIA- 603 102.

Abstract

The states $|\alpha, m\rangle$, defined as $\hat{a}^{\dagger m}|\alpha\rangle$ up to a normalization constant and m is a nonnegative integer, are shown to be the eigenstates of $f(\hat{n}, m)\hat{a}$ where $f(\hat{n}, m)$ is a nonlinear function of the number operator $\hat{n}$. The explicit form of $f(\hat{n}, m)$ is constructed. The eigenstates of this operator for negative values of m are introduced. The properties of these states are discussed and compared with those of the state $|\alpha, m\rangle$.

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I. INTRODUCTION

Coherent states are important in many fields of physics [1,2]. Coherent states $|\alpha\rangle$, defined as the eigenstates of the harmonic oscillator annihilation operator $\hat{a}$, $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ [3], have properties like the classical radiation field. There exist states of the electromagnetic field whose properties, like squeezing, higher order squeezing, antibunching and sub-Poissonian statistics [4,5], are strictly quantum mechanical in nature. These states are called as non-classical states. The coherent states define the limit between the classical and nonclassical behaviour of the radiation field as far as the nonclassical effects are considered. A generalization of the coherent states was done by q-deforming the basic commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$ [6,7]. A further generalization is to define states that are eigenstates of the operator $f(\hat{n})\hat{a}$,

$$f(\hat{n})\hat{a}|f, \alpha\rangle = \alpha|f, \alpha\rangle, \tag{1}$$

where $f(\hat{n})$ is a operator valued funtion of the number operator $\hat{n} = \hat{a}^\dagger\hat{a}$. These eigenstates are called as nonlinear coherent states and they are nonclassical. In the linear limit, $f(\hat{n}) = 1$, the nonlinear coherent states become the usual coherent states $|\alpha\rangle$. The nonlinear coherent states were introduced, as f-coherent states, in connection with the study of the oscillator whose frequency depends on its energy [8]. A class of nonlinear coherent states can be realized physically as the stationary states of the center-of-mass motion of a trapped ion [9]. These nonlinear coherent states exhibit nonclassical features like squeezing and self-splitting.

The photon-added coherent states $|\alpha, m\rangle$ [10] are defined as

$$|\alpha, m\rangle = \frac{\hat{a}^{\dagger m}|\alpha\rangle}{\sqrt{\langle\alpha|\hat{a}^m\hat{a}^{\dagger m}|\alpha\rangle}}, \tag{2}$$

where m is a nonegative integer. The states $|\alpha, m\rangle$ exhibit nonclassical features like phase squeezing and sub-Poissonian statistics. These states are produced in the interaction of a two-level atom with a cavity field initially prepared in the coherent state $|\alpha\rangle$ [10]. In the present contribution it is shown that the photon-added coherent states can be interpreted as nonlinear coherent states. This is done by showing that the states $|\alpha, m\rangle$ obey the equation

$$f(\hat{n},m)\hat{a}|\alpha,m\rangle = \alpha|\alpha,m\rangle, \tag{3}$$

with a suitable choice for the function $f(\hat{n},m)$. The operator $f(\hat{n},m)$ is a valid operator for negative values of m also. The eigenstates of $f(\hat{n},m)\hat{a}$ with negative values of m are constructed and their properties compared with those of $|\alpha,m\rangle$.

II. CONSTRUCTION OF $f(\hat{n},m)$

In this section we construct the explicit form of the operator valued function $f(\hat{n},m)$. The coherent states $|\alpha\rangle$ satisfy, by definition,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \tag{4}$$

where α is a complex number. Premultiplying both the sides of this equation by $\hat{a}^{\dagger m}$ yields

$$\hat{a}^{\dagger m}\hat{a}|\alpha\rangle = \alpha\hat{a}^{\dagger m}|\alpha\rangle, \tag{5}$$

where m is a nonnegative integer. Using the commutation relation $[\hat{a},\hat{a}^{\dagger}] = 1$, the above equation is written as

$$(\hat{a}\hat{a}^{\dagger m} - m\hat{a}^{\dagger(m-1)})|\alpha\rangle = \alpha\hat{a}^{\dagger m}|\alpha\rangle, \tag{6}$$

which, making use of the identity $\frac{1}{1+\hat{a}^{\dagger}\hat{a}}\hat{a}\hat{a}^{\dagger} = 1$, leads to

$$(\hat{a} - \frac{m}{1+\hat{a}^{\dagger}\hat{a}}\hat{a})\hat{a}^{\dagger m}|\alpha\rangle = \alpha\hat{a}^{\dagger m}|\alpha\rangle. \tag{7}$$

Comparing Eq. (7) and Eq. (3) gives the expression for $f(\hat{n},m)$ as

$$f(\hat{n},m) = 1 - \frac{m}{1+\hat{a}^{\dagger}\hat{a}}. \tag{8}$$

This shows that the photon-added coherent states can be interpreted as nonlinear coherent states. These states are a class nonlinear cohrent states than can be physically realized in the interaction of a two- level atom with a cavity field that is initially prepared in the cohrent state $|\alpha\rangle$.

III. $|\alpha, m\rangle$ AS DEFORMED m-PHOTON NUMBER STATE $|m\rangle$

In this section it is shown that the photon-added coherent states can be written as the nonunitarily deformed number state . This is achieved by the method given by Shanta, et al [11]. Firstly, a brief review of the method is given. Consider an "annihilation operator" $\hat{A}$ which annihilates a set of number states $|n_i\rangle, i = 1, 2, ...k$. Then we can construct a sector S_i by repeatedly applying $\hat{A}^\dagger$, the adjoint of $\hat{A}$, on the number state $|n_i\rangle$. Thus we have k sectors corresponding to the states that are annihilated by $\hat{A}$. A given sector may turnout to be either finite or infinite dimensional. If a sector, say S_j, is of infinite dimension then we construct an operator $\hat{G_j}^\dagger$ such that the commutator $[\hat{A}, \hat{G_j}^\dagger] = 1$ holds in the sector. Then the eigenstates of $\hat{A}$ can be written as $e^{\alpha \hat{G_j}^\dagger}|n_j\rangle$. If the operator $\hat{A}$ is of the form $f(\hat{n})\hat{a}^p$, where p is a nonnegative integer and $f(\hat{n})$ is a operator valued function of the number operator $\hat{a}^\dagger\hat{a}$, such that it annihilates the number state $|j\rangle$ then $\hat{G_j}^\dagger$ is constructed as

$$\hat{G_j}^\dagger = \frac{1}{p}\hat{A}^\dagger \frac{1}{\hat{A}\hat{A}^\dagger}(\hat{a}^\dagger\hat{a} + p - j). \tag{9}$$

The photon-added coherent states are the eigenstates of $f(\hat{n}, m)\hat{a}$ with $f(\hat{n}, m)$ given by Eq.(8). This operator annihilates the vacuum state $|0\rangle$ and the m-photon state $|m\rangle$. The states in between the vacuum state and the m-photon state are not annihilated. In this sense it is different from the m-photon annihilation operator $\hat{a}^m$ which annihilates all the states $|i\rangle$, $i = 0, 1, 2, ..m$. To discuss the case of $|\alpha, m\rangle$ let

$$\hat{A} = (1 - \frac{m}{1 + \hat{a}^\dagger\hat{a}})\hat{a}. \tag{10}$$

The adjoint $\hat{A}^\dagger$ is given by

$$\hat{A}^\dagger = \hat{a}^\dagger(1 - \frac{m}{1 + \hat{a}^\dagger\hat{a}}). \tag{11}$$

We construct the sector S_0 by repeatedly applying $\hat{A}^\dagger$ on the vacuum state $|0\rangle$. S_0 is the set $|i\rangle, i = 0, 1, 2, ...m - 1$ and it is finite dimensional. The secotr S_m, built by the repeated application of $\hat{A}^\dagger$ on $|m\rangle$, is the set $|i\rangle, i = m, m + 1, ...$ and it is of infinite dimension. Hence we can construct an operator $\hat{G}^\dagger$ such that $[\hat{A}, \hat{G}^\dagger] = 1$ holds in S_m. To construct $\hat{G}^\dagger$,

corresponding to the operator $\hat{A}$ given by Eq.(10), we set $p = 1$ and $j = m$ in Eq.(9) and this yields

$$\hat{G}^\dagger = \hat{a}^\dagger. \tag{12}$$

Thus on the sector S_m we have $[\hat{A}, \hat{a}^\dagger] = 1$. Thus the photon-added coherent states, which are the eigenstates of $\hat{A}$, can be written as $e^{\alpha\hat{a}^\dagger}|m\rangle$. However, this is not a unitary deformation.

IV. EIGENSTATES OF $f(\hat{n}, m)\hat{a}$ WITH NEGATIVE m

The form of $f(\hat{n}, m)$, given by Eq. (8), suggests that it is a well defined operator valued function, on the harmonic oscillator Hilbert space, for negative integer values of m also. In this section the nonlinear coherent states, with negative m in the expression for $f(\hat{n}, m)$, are constructed. Denoting the eigenstates by $|\alpha, -m\rangle$, the equation to determine them is

$$(1 + \frac{m}{1 + \hat{a}^\dagger\hat{a}})\hat{a}|\alpha, -m\rangle = \alpha|\alpha, -m\rangle. \tag{13}$$

Expanding $|\alpha, -m\rangle$ in terms of the number states $|n\rangle$ as

$$|\alpha, -m\rangle = \sum_{n=0}^{\infty} c_n|n\rangle, \tag{14}$$

where c_n's are the expansion coefficients and substituting the expansion in Eq. (13) leads to the recursion relation

$$c_n = \frac{m!\sqrt{n!}\alpha^n}{(n+m)!}c_0. \tag{15}$$

The constant c_0 is determined by normalization. The normalized $|\alpha, -m\rangle$ is given by

$$|\alpha, -m\rangle = N \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}(n+1)(n+2)...(n+m)}|n\rangle; \tag{16}$$

$$N^{-1} = \sqrt{\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}n!}{(n+m)!^2}} = \frac{1}{m!}\sqrt{{}_2F_2(1, 1, m+1, m+1, |\alpha|^2)},$$

where ${}_2F_2(1, 1, m+1, m+1, |\alpha|^2)$ is the generalized Hypergeometric function [12]. The number state expansion for the state $|\alpha, m\rangle$ is [10]

$$|\alpha, m\rangle = \frac{\exp(-|\alpha|^2/2)}{\sqrt{L_m(-|\alpha|^2)m!}} \sum_{n=0}^{\infty} \frac{\alpha^n \sqrt{(m+n)!}}{n!} |n+m\rangle, \tag{17}$$

where $L_m(x)$ is the Lauguerre polynomial of order m defined by [12]

$$L_m(x) = \sum_{n=0}^{m} \frac{(-x)^n m!}{(n!)^2 (m-n)!}. \tag{18}$$

The state $|\alpha, -m\rangle$ involves a superpostion of all the Fock states starting with the vacuum state $|0\rangle$. But in the state $|\alpha, m\rangle$ the Fock states $|0\rangle, |1\rangle ... |m-1\rangle$ are not present. This important difference leads to different limiting cases of the states $|\alpha, m\rangle$ and $|\alpha, -m\rangle$ when $\alpha \to 0$. In the limit $\alpha \to 0$ the state $|\alpha, -m\rangle$ becomes the vacuum state $|0\rangle$, which has properties like a classical radiation field, irrespective of the value of m and the state $|\alpha, m\rangle$ becomes the Fock state $|m\rangle$ which is nonclassical. In the limit $m \to 0$ the states $|\alpha, m\rangle$ and $|\alpha, -m\rangle$ become the coherent state $|\alpha\rangle$. Thus, $|\alpha, -m\rangle(|\alpha, m\rangle)$ is a state that is intermediate between the vacuum state (the number state $|m\rangle$) and the coherent state.

The photon-added coherent states are obtained by the action of $\hat{a}^{\dagger m}$ on the coherent state. The states $|\alpha, -m\rangle$ can be written in a similar form using the inverse operators $\hat{a}^{-1}$ and $\hat{a}^{\dagger -1}$ [13]. These operators are defined in terms of their actions on the number states $|n\rangle$ as follows:

$$\hat{a}^{-1}|n\rangle = \frac{1}{\sqrt{n+1}}|n+1\rangle, \tag{19}$$

$$\hat{a}^{\dagger -1}|n\rangle = \frac{1}{\sqrt{n}}|n-1\rangle \quad \text{for n} \neq 0, \tag{20}$$

$$\hat{a}^{\dagger -1}|0\rangle = 0. \tag{21}$$

Using these inverse operators and Eqn.(16) the state $|\alpha, -m\rangle$ can be written as

$$|\alpha, -m\rangle = N\hat{a}^{\dagger -m}\hat{a}^{-m}|\alpha\rangle. \tag{22}$$

The states $|\alpha, -m\rangle$ correspond to the nonlinear coherent states with -m replacing m in $f(\hat{n}, m)$. However, they are obtained by the action of $\hat{a}^{\dagger -m}\hat{a}^{-m}$ on $|\alpha\rangle$ and not $\hat{a}^{\dagger -m}$ on $|\alpha\rangle$. Using the method reviewed in Section III we can show that

$$|\alpha, -m\rangle = e^{\alpha \hat{G}^\dagger}|0\rangle, \tag{23}$$

where $\hat{G}^\dagger = \hat{a}^\dagger \frac{1+\hat{a}^\dagger \hat{a}}{1+m+\hat{a}^\dagger \hat{a}}$.

V. SQUEEZING IN $|\alpha, -m\rangle$

The state $|\alpha, -m\rangle$ exhibits squeezing in both x- and p-quadratures. The x- and p-quadratures are given interms of $\hat{a}$ and $\hat{a}^\dagger$ by

$$x = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}}, \quad p = \frac{\hat{a} - \hat{a}^\dagger}{i\sqrt{2}}. \tag{24}$$

The mean values of the relevant operators in the state $|\alpha, -m\rangle$ are

$$\langle \hat{a} \rangle = \alpha N^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n} n!}{(n+m)!^2} \frac{(n+1)}{(n+m+1)}, \tag{25}$$

$$\langle \hat{a}^2 \rangle = \alpha^2 N^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n} n!}{(n+m)!^2} \frac{(n+1)(n+2)}{(n+m+1)(n+m+2)}, \tag{26}$$

and

$$\langle \hat{a}^\dagger \hat{a} \rangle = N^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n} n!}{(n+m)!^2} n. \tag{27}$$

The mean values of $\hat{a}^\dagger$ and $\hat{a}^{\dagger 2}$ are obtained by taking the complex conjugates of $\langle \hat{a} \rangle$ and $\langle \hat{a}^2 \rangle$ respectively. The uncertainty in x is

$$\langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{2}[1 + 2\langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a}^2 \rangle + \langle \hat{a}^{\dagger 2} \rangle - \langle \hat{a} \rangle^2 - \langle \hat{a}^\dagger \rangle^2 - 2\langle \hat{a} \rangle \langle \hat{a}^\dagger \rangle], \tag{28}$$

and that in p is

$$\langle p^2 \rangle - \langle p \rangle^2 = \frac{1}{2}[1 + 2\langle \hat{a}^\dagger \hat{a} \rangle - \langle \hat{a}^2 \rangle - \langle \hat{a}^{\dagger 2} \rangle + \langle \hat{a} \rangle^2 + \langle \hat{a}^\dagger \rangle^2 - 2\langle \hat{a} \rangle \langle \hat{a}^\dagger \rangle]. \tag{29}$$

In Fig. 1 the variance in p is shown for real α for various values of m. As expected the uncertainty in p is close to $\frac{1}{2}$, the unertainty in p in the vacuum state, when α is close to zero. In the case of the state $|\alpha, m\rangle$ the variance is close to $m + \frac{1}{2}$ when α is close to zero. For real values of α the p-quadrature is always squeezed, $\langle p^2 \rangle - \langle p \rangle^2 < \frac{1}{2}$, for the state $|\alpha, -m\rangle$. For large values of α the variance in p approaches that of the coherent state. If α becomes $i\alpha$ the expression for variance in p becomes the expression for variance in x. Since p shows squeezing for real α the x-quadrature exhibits squeezing for imaginary α.

VI. PHOTON STATISTICS OF $|\alpha, -m\rangle$

The photon number distribution $p(n)$ for the state $|\alpha, -m\rangle$ is

$$\begin{aligned} p(n) &= |\langle n|\alpha, -m\rangle|^2, \\ &= N\frac{|\alpha|^{2n} n!}{(n+m)!^2}. \end{aligned} \quad (30)$$

When $m = 0$ the distribution becomes the Poissonian distribution whose mean value is $|\alpha|^2$.

A measure of the variance of the photon number distribution is given by the Mandel's q parameter [14],

$$q = \frac{\langle(\triangle\hat{n})^2\rangle - \langle\hat{n}\rangle}{\langle\hat{n}\rangle}. \quad (31)$$

The coherent states have q as zero. Negative value of q indicates that the distribution $p(n)$ is sub-Poissonian and it is a nonclassical feature. The photon-added coherent states $|\alpha, m\rangle$ are always sub-Poissonian for all values of m. For the state $|\alpha, -m\rangle$ the mean values of $\hat{n}$ and $\hat{n}^2$ are given by

$$\langle\hat{n}\rangle = N^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n} n!}{(n+m)!^2} n, \quad (32)$$

$$\langle\hat{n}^2\rangle = N^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n} n!}{(n+m)!^2} n^2. \quad (33)$$

In Fig. 2 the q-parameter, calculated from Eqs. (31)-(33), for the state $|\alpha, -m\rangle$ is shown as a function of $|\alpha|$. The states $|\alpha, -m\rangle$ have q always greater than zero indicating that they are super-Poissonian. For small values of α the q-parameter is close to zero and for large values of α it approaches zero.

VII. SUMMARY

The photon-added coherent states are nonlinear coherent states. They are the eigenstates of the operator $(1 - \frac{m}{1+\hat{a}^\dagger\hat{a}})\hat{a}$ when m takes positive integer values. This operator is a meaningful operator when m takes negative integer values also. The corresponding eigenstates $|\alpha, -m\rangle$ are nonclassical. The photon-added coherent state $|\alpha, m\rangle$ results from the

action of $\hat{a}^{\dagger m}$ on the coherent state $|\alpha\rangle$ while the state $|\alpha, -m\rangle$ comes from the action of $\hat{a}^{\dagger -m}\hat{a}^{-m}$ on the coherent state $|\alpha\rangle$. Both $|\alpha, m\rangle$ and $|\alpha, -m\rangle$ show squeezing. While $|\alpha, m\rangle$ is sub-Poissonian the state $|\alpha, -m\rangle$ is not. The states $|\alpha, m\rangle$ and $|\alpha, -m\rangle$ become $|m\rangle$ and $|0\rangle$ respectively in the limit $\alpha \to 0$ but they become the coherent state $|\alpha\rangle$ when $m \to 0$. The states $|\alpha, m\rangle$ and $|\alpha, -m\rangle$ are the result of deforming the number states $|m\rangle$ and $|0\rangle$ respectively.

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REFERENCES

[1] J.R. Klauder and B.-S. Skagerstam, *Coherent States- Applications in Physics and Mathematical Physics (World Scientific, Singapore 1985).*

[2] W.-M. Zhang, D.H. Feng and R. Gilmore, Rev. Mod. Phys. **62**, 867 (1990).

[3] R.J. Galuber, Phys. Rev. **130**, 2529 (1963); **131**, 2766 (1963); Phys. Rev. Lett. **10**, 84 (1963).

[4] D.F. Walls, Nature(London), **306**, 141 (1983).

[5] R. Loudon and P.L. Knight, J. Mod. Opt. **34**, 709 (1987).

[6] L.C. Biedenharn J.Phys.A **22**, L873 (1989).

[7] A.J. Macfarlane, J. Phys. A **22**, 4581 (1989).

[8] V.I. Man'ko, G.Marmo, F. Zaccaria and E.C.G. Sudarshan in Proceedings of the IV Wigner Symposium, edited by N. Atakishiyev, T.Seligman and K.B.Wolf (World Scientific, Singapore, 1996).

[9] R.L. de Matos Filho and W. Vogel, Phys.rev.A **54**, 4560 (1996).

[10] G.S. Agarwal and K. Tara, Phys.Rev.A **43**,492 (1991).

[11] P. Shanta, S. Chaturvedi, V.Srinivasan, G.S.Agarwal and C.L. Mehta, Phys. Rev. Lett. **72**, 1447 (1994).

[12] I.S. Gradshteyn and I.M. Ryzhik, Tables of Integrals, Series and Products (Academic Press, London 1994) p. 1071.

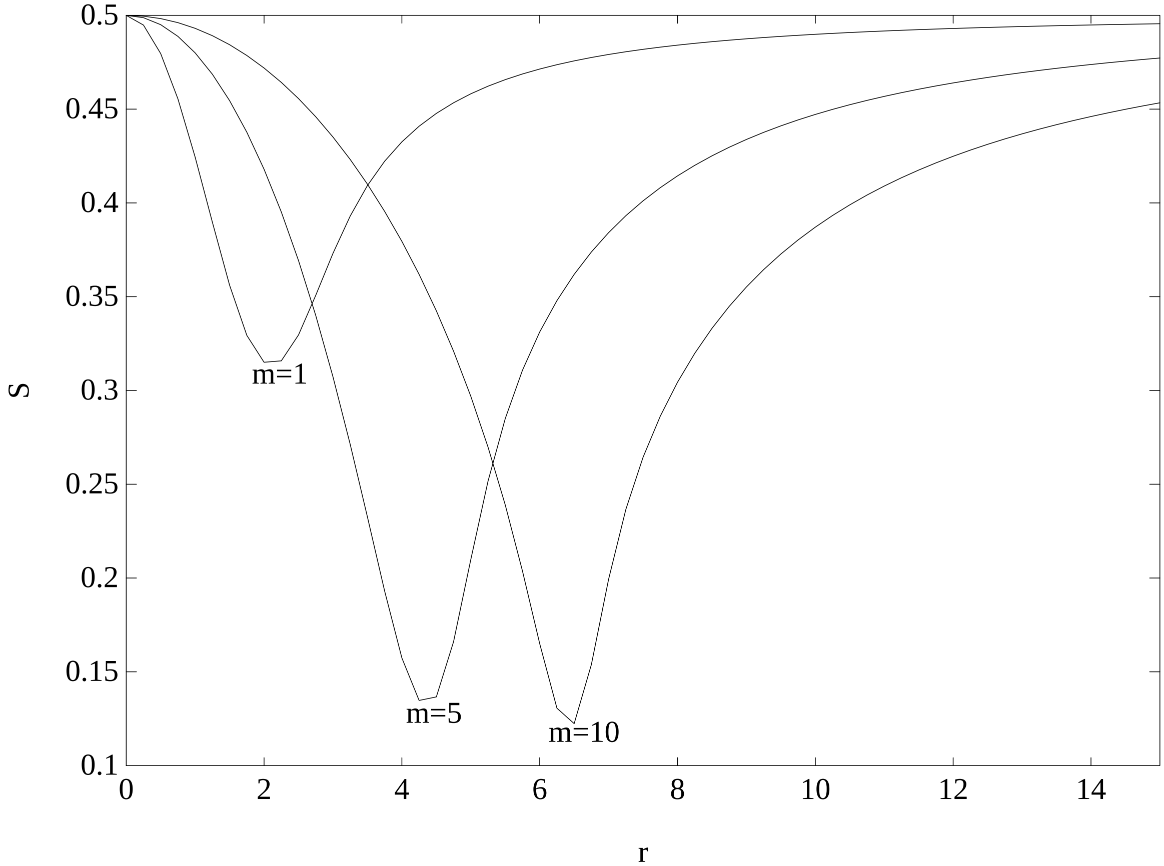
[13] C.L. Mehta, Anil K. Roy and G.M. Saxena, Phys.Rev.A **46** 1565 (1992).

[14] L.Mandel, Opt. Lett. **4**, 205(1979).

FIGURES

FIG. 1. Uncertainty S, $\langle p^2 \rangle - \langle p \rangle^2$, in p as a function of α for m=1, m=5 and m=10 for the state $|\alpha, -m\rangle$. The real α is denoted as r.

FIG. 2. Mandel's q parameter as a function of $|\alpha|$ for m = 1, m = 5 and m = 10. $|\alpha|$ is denoted as r


0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
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0
2
4
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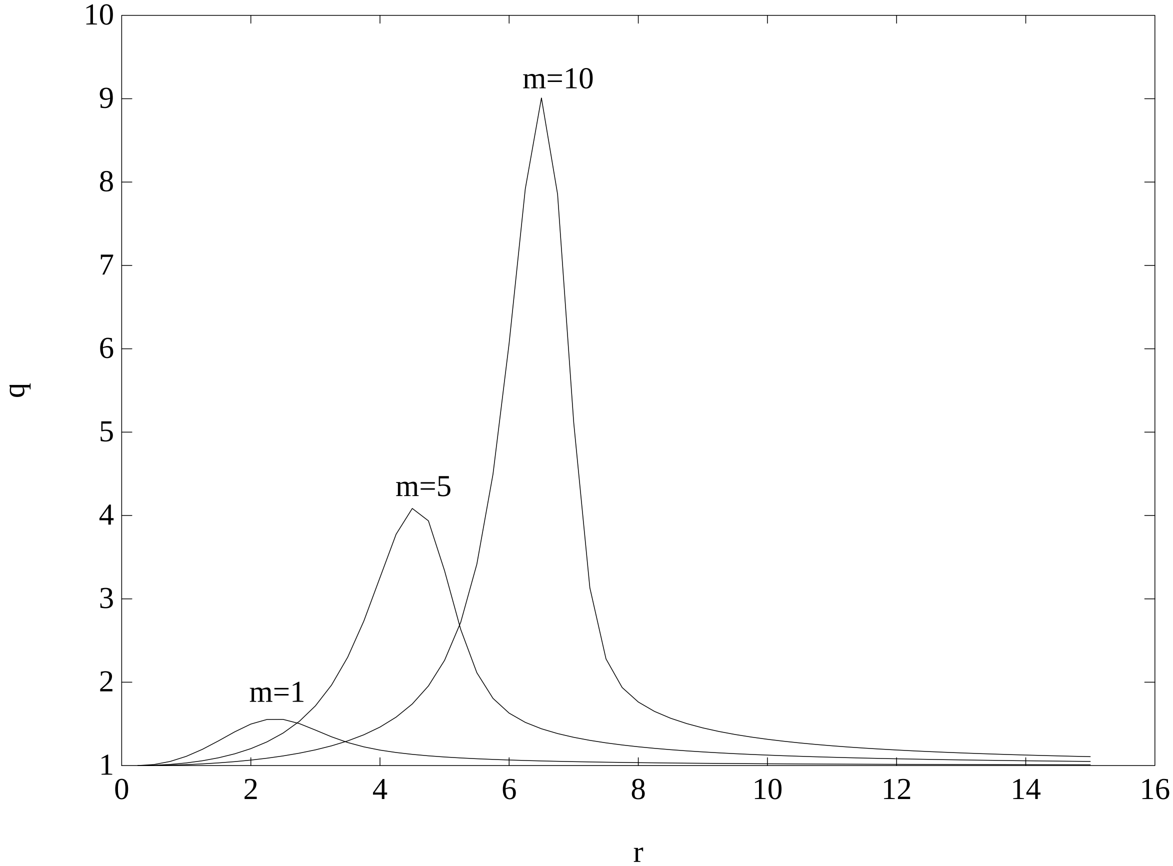
14
r
m=1
m=5
m=10



m=10
m=5
m=1
q
r